On the Realignment Criterion and Beyond

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Summary

- separability problem: determine if a quantum states of a composite system is separable or entangled
  - the Realignment Criterion (RC)
- extensions and generalizations of the RC
  - conjectures and examples, but no theorem yet
- application to the study of quantum channels
  - entanglement needs space
- open questions
  - looking for examples and theorems

Motivations

- Entanglement as a resource for Quantum Information Tasks
  - cryptography
  - teleportation
  - computation
- Entanglement in many body systems
  - phase transition
  - quantum simulation
- Entanglement in mixed states
  - experimental errors
  - distillability
  - free entanglement vs bound entanglement

Intro

Entanglement vs Separability (for discrete variables) (1)

We consider bi-partite systems

$$S \equiv S_A \times S_B \quad \longleftrightarrow \quad \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{1}$$

For simplicity, we consider discrete symmetric systems

$$\mathcal{H}_A \cong \mathcal{H}_B \cong \mathbb{C}^d \tag{2}$$

There are states that can be prepared with only local operations and classical communications (Separable states)

$$\rho = \sum_k p_k \rho_A^{(k)} \otimes \rho_B^{(k)} \qquad p_k \geq 0 \, , \quad \sum_k p_k = 1 \tag{3}$$

(Two experimenters, that can manipulate only one of the two subsystems, share a common random number generator. In correspondence of the output k (which has probability p_k), they locally prepare $\rho_A^{(k)}$ and $\rho_B^{(k)}$.)

Intro

Entanglement vs Separability (for discrete variables) (3)

If A and B are allowed to interact directly, there are many more possibilities.

Entangled states cannot be written as statistical ensemble over local states:

$$\rho \neq \sum_k p_k \rho_A^{(k)} \otimes \rho_B^{(k)} \tag{4}$$

Entangled states are defined as not-separable states.

How can entangled states be characterized?

A Separability Criterion is a necessary condition for a quantum state ρ:

$$\rho \ \text{sep} \quad \Rightarrow \quad F(\rho) \geq 0 \tag{5}$$

- Positive Partial Transpose (PPT) criterion [Peres PRL, Horodecki's PLA (1996)]

$$\rho \ \text{sep} \quad \Rightarrow \quad \rho^{T_A} \geq 0 \ \ (\rho^{T_B} \geq 0) \tag{6}$$

- Realignment Criterion (RC) [Chen et al. Quant. Inf. Comp. (2003)]

$$\rho \ \text{sep} \quad \Rightarrow \quad \mathrm{tr}|\tilde{\rho}| \leq 1 \tag{7}$$

They can be understood in a unified way (Linear Contraction Approach [Horodecki's Open Sys. Inf. Dyn. (2006)]).

Suppose that an entangled state ρ is prepared between systems (particles, modes of the e.m.f., etc.) A and B.

Suppose also that system B is send to a remote location (through a optic fiber, for instance)... the transmission is in general noisy:

$$\rho \quad \longrightarrow \quad \rho' = (\mathcal{I}_A \otimes \mathcal{L})(\rho) \tag{8}$$

General considerations yields that $\mathcal{L}$ is a Completely Positive (CP) map [Sudarshan et al. PR (1961)].

If we look entanglement as a physical resource, one is interested in preserve entanglement at the end of the channel.

This leads to the definition of Entanglement Breaking (EB) channels [Horodecki et al. Rev. Math. Phys. (2003)]:

$$\forall \rho \qquad \rho' = (\mathcal{I}_A \otimes \mathcal{L})(\rho) \qquad \text{sep} \tag{9}$$

How to characterize EB channels?

There is a correspondence between quantum states and quantum channels [Verstraete et al. quant-ph/0202124, Jamiolkowski Rep. Math. Phys. (1972)]...

$$\rho = (\mathcal{I} \otimes \mathcal{L})(\beta) \tag{10}$$

where $\beta = |I\rangle\langle I|$, with $|I\rangle = \sum_{i=1}^{N} |ii\rangle$ is an unnormalized maximally entangled state.

In terms of components, they are $d^2 \times d^2$ matrices:

$$\mathcal{L} = \rho_{i\alpha j\beta} |\alpha\beta\rangle\langle ij| \tag{11}$$

if

$$\rho = \rho_{i\alpha j\beta} |i\alpha\rangle\langle j\beta| \tag{12}$$

(corresponds to a realignment of the matrix elements)

Realignment Criterion (1)

To each quantum state we can associate a realigned operator

$$\rho \quad \longleftrightarrow \quad \tilde{\rho} \tag{13}$$

$$\rho = \rho_{i\alpha j\beta} |i\alpha\rangle\langle j\beta| \tag{14}$$

and

$$\tilde{\rho} = \rho_{i\alpha j\beta} |\alpha\beta\rangle\langle ij| \tag{15}$$

Realignment Criterion (2)

We can consider the SVD of the realigned matrix:

$$\tilde{\rho} = U\Lambda V^{\dagger} \qquad \Lambda = \mathrm{diag}\{\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots \tilde{\lambda}_{d^2}\} \tag{16}$$

The RC states that:

$$\rho \ \text{sep} \ \Rightarrow \ \sum_i \tilde{\lambda}_i \leq 1 \tag{17}$$

or, in other words:

$$\rho \ \text{sep} \ \Rightarrow \ \mathrm{tr}|\tilde{\rho}| \leq 1 \tag{18}$$

A family of Criteria (1)

The RC deals with the trace of the realigned matrix:

$$\mathrm{tr}|\tilde{\rho}| = \sum_{k=1}^{d^2} \tilde{\lambda}_k \tag{19}$$

...we can also consider its determinant:

$$\det|\tilde{\rho}| = \prod_{k=1}^{d^2} \tilde{\lambda}_k \tag{20}$$

...or the sum of the principal minors of order l:

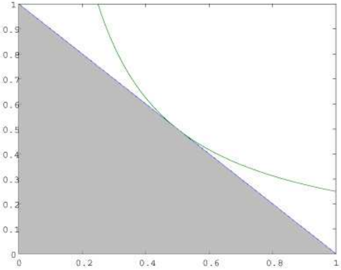
$$M^{[l]}(|\tilde{\rho}|) = \sum_{\{i_i, i_2, \dots i_l\}} \tilde{\lambda}_{i_1} \tilde{\lambda}_{i_2} \dots \tilde{\lambda}_{i_l} \tag{21}$$

A family of Criteria (2)

At a first sight, we can write the following chain of implications:

$$\rho \text{ sep} \quad \Rightarrow \quad \mathrm{tr}|\tilde{\rho}| \leq 1 \quad \Rightarrow \quad \det|\tilde{\rho}| \leq \left(\frac{1}{d^2}\right)^{d^2} \tag{22}$$

$$\rho \text{ sep} \quad \Rightarrow \quad M^{[l]}(\tilde{\rho}) \leq \binom{d^2}{l}\left(\frac{1}{d^2}\right)^{l} \tag{23}$$



Application to the study of Quantum Channels

By means of the duality relation between states and maps, separable states correspond to EB quantum channels [Holevo quant-ph/980923]

Thus we have the following criterion for a channel to be EB:

$$\mathcal{L} \ \text{ent breaking} \quad \Rightarrow \quad |\det \mathcal{L}| \leq \left(\frac{1}{d^2}\right)^{d^2} \qquad (24)$$

Notice that the $\det$ of a linear map $\mathcal{L}$ is interpreted as the volume of the ellipsoid in which the unit sphere is mapped by $\mathcal{L}$.

Entanglement needs space (!)

Degenerate case

Analogously, we can consider the minors:

$$\mathcal{L} \text{ ent breaking } \Rightarrow M^{[l]}(\mathcal{L}) \leq \binom{d^2}{l} \left(\frac{1}{d^2}\right)^l \tag{25}$$

...in the case the map is degenerate, with rank r, we obtain more stringent conditions:

$$\mathcal{L} \text{ ent breaking } \Rightarrow M^{[l]}(\mathcal{L}) \leq \binom{r}{l} \left(\frac{1}{r}\right)^l \tag{26}$$

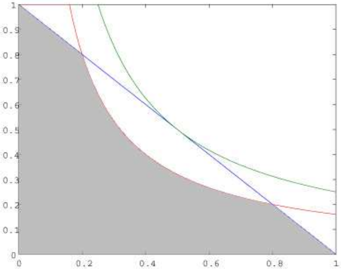
Generalizations of the Criteria

Can we find a lower upper bound?

...for instance can we prove that:

$$\rho \text{ sep} \quad \Rightarrow \quad \det|\tilde{\rho}| \leq x < \left(\frac{1}{d^2}\right)^{d^2} \tag{27}$$

This would correspond to restrict the region of entangled states that are recognized by the RC.



A lower upper bound exists (1)

Proof: Let us suppose the existence of a *separable* density matrix ρ_0 such that $\det(|\rho_0|) = d^{-2d^2}$. This implies that $\tilde{\lambda}_k = d^{-2}$. Hence the SVD is:

$$\tilde{\rho}_0 = d^{-2} U V^\dagger, \qquad (28)$$

where U and V have matrix elements $u_{(\alpha\beta)(\alpha'\beta')}$ and $v_{(ij)(i'j')}$. So we have:

$$u_{(\alpha\beta)(kl)} {v^*}_{(kl)(ij)} = \langle \hat{v}_{ij}, \hat{u}_{\alpha\beta} \rangle \qquad (29)$$

where $\hat{u}_{\alpha\beta}$ and $\hat{v}_{ij}$ are respectively a row and a column of the matrices U and V.

A lower upper bound exists (2)

Since the corresponding ρ_0 is a well defined density operator we have $\mathrm{tr}(\rho_0) = 1$. We have:

$$\mathrm{tr}(\rho_0) = d^{-2} \sum_{i,\alpha} \langle \hat{v}_{ii}, \hat{u}_{\alpha\alpha} \rangle = d^{-2} \langle \sum_i \hat{v}_{ii}, \sum_\alpha \hat{u}_{\alpha\alpha} \rangle \tag{30}$$

and we obtain:

$$\begin{aligned} \mathrm{tr}(\rho_0) = |\mathrm{tr}(\rho_0)| = d^{-2} |\langle \sum_i \hat{v}_{ii}, \sum_\alpha \hat{u}_{\alpha\alpha} \rangle| \leq \\ d^{-2} |\sum_i \hat{v}_{ii}| |\sum_\alpha \hat{u}_{\alpha\alpha}| = d^{-1} \end{aligned} \tag{31}$$

which is in contradiction with the hypothesis that ρ_0 is a well defined density operator. Since the set of separable states is compact, a lower upper bound must exists. $\square$

Find the lower upper bounds

To find the value of the lower upper bounds can be a rather difficult problem, since determinants and minors do not behave friendly with the convex structure of separable state.

For *qubit-qubit* and *qubit-qutrit* systems we can search numerically their value

	tr	$M^{[2]}$	$M^{[3]}$	det
RC	1	0.375	0.0625	0.0039
bRC	1	0.3333	0.0463	0.0023

	tr	$M^{[2]}$	$M^{[3]}$	det
RC	1	0.375	0.0625	0.0039
bRC	1	0.3469	0.0525	0.0029

Open Question (1)

Are these criteria independent of the parent RC?

Open Question (2)

$$\mathcal{L} \text{ ent breaking } \Rightarrow M^{[l]}(\mathcal{L}) \leq x(l,d) < \binom{d^2}{l}\left(\frac{1}{d^2}\right)^l \qquad (32)$$

What is the value of $x(l,d)$??

Conclusion and Outlook

- starting from the RC, a whole family of weaker separability criteria can be derived
- these criteria can be applied also to study of quantum channels and yield to a relation between entanglement and geometry
- a generalization of the criteria seems to be possible which can be stronger than the parent RC, but for interesting applications we need a theorem (!)
- at least for low dimensional systems we are looking for examples in which the proposed criteria are stronger than the RC